

## Investigation of Geometric Characteristics of Two Subclasses of $(\alpha, \beta)$ -Metrics

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### ABSTRACT

In this paper, we find the necessary and sufficient condition to characterize the projective relation between two subclasses of  $(\alpha, \beta)$ -metrics  $L = \alpha + \beta - \frac{\beta^2}{\alpha}$  and  $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$  on a manifold  $M$  with dimension  $n \geq 3$ , where  $\alpha$  and  $\bar{\alpha}$  are two Riemannian metrics,  $\beta$  and  $\bar{\beta}$  are two non-zero 1-forms.

**Keywords:** Finsler space,  $(\alpha, \beta)$ metric, Kropina metric, Projective change, Douglas metric.

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### I. INTRODUCTION

In Finsler geometry, two Finsler metrics  $F$  and  $\bar{F}$  on a manifold  $M$  are called projectively related if  $G^i = \bar{G}^i + Py^i$ , where  $G^i$  and  $\bar{G}^i$  are the geodesic coefficients of  $F$  and  $\bar{F}$  respectively and  $P = P(x, y)$  is a scalar function on the slit tangent bundle  $TM_0$ . In this case, any geodesic of the first is also geodesic for the second and viceversa. The projective changes between two Finsler spaces have been studied by [1], [2], [3], [4], [6], [11],[13],[14], [18], [19], [20].

$(\alpha, \beta)$ -metrics form a special and very important classes of Finsler metrics which can be expressed in the for  $F = \alpha\varphi(s)$ :  $s = \frac{\beta}{\alpha}$ , where  $\alpha$  is a Riemannian metric and  $\beta$  is a 1-form and  $\varphi$  is a  $C^\infty$  positive function on the definite domain. In particular, when  $\varphi = \frac{1}{s}$ , the Finsler metric  $F = \frac{\beta^2}{\alpha}$  is called Kropina metric. Kropina metric was first introduced by L. Berwald in connection with two dimensional Finsler space with rectilinear extremal and was investigated by V.K. Kropina [7]. They together with Randers metric are C-reducible [10]. However, Randers metric are regular Finsler metric but Kropina metric is non-regular Finsler metric. Kropina metric seem to be among the simplest nontrivial Finsler metric with many interesting applications in physics, electron optics with a magnetic field, dissipative mechanics and irreversible thermodynamics [5], [15]. Also, there are interesting applications in relativistic field theory, evolution and developmental biology.

Based on Stavrino's work on Finslerian structure of anisotropic gravitational field [16], we know that the anisotropy is an issue of the background radiation for all possible  $(\alpha, \beta)$ -metrics. Then the 1-form  $\beta$  represents the same direction of the observed anisotropy of the microwave background radiation. That is, if two  $(\alpha, \beta)$ -metrics  $F = \alpha\varphi\left(\frac{\beta}{\alpha}\right)$  and  $\bar{F} = \bar{\alpha}\bar{\varphi}\left(\frac{\bar{\beta}}{\bar{\alpha}}\right)$  are the same anisotropy directions (or, they have the same axis rotation to their indicatrices), then their 1-form.

$\beta$  and  $\bar{\beta}$  are collinear, there is a function  $\mu \in C^\infty(M)$  such that  $\beta(x, y) = \mu\bar{\beta}(x, y)$ . By [3], for the projective equivalence between a general  $(\alpha, \beta)$ -metric and a Kropina metric, we have the following lemma:

**Lemma 1.1.** Let  $F = \alpha\varphi\left(\frac{\beta}{\alpha}\right)$  be an  $(\alpha, \beta)$ -metric on  $n$ -dimensional manifold  $M(n \geq 3)$ , satisfying that  $\beta$  is not parallel with respect to  $\alpha$ ,  $db \neq 0$  everywhere (or)  $b = \text{constant}$  and  $F$  is not of Randers type. Let  $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$  be a Kropina metric on the manifold  $M$ , where  $\bar{\alpha} = \lambda(x)\alpha$  and  $\bar{\beta} = \mu(x)\beta$ . Then  $F$  is Projectively Equivalent to  $\bar{F}$  if and only if the following equations holds,

$$[1 + (k_1 + k_2s^2)s^2 + k_3s^2]\varphi'' = (k_1 + k_2s^2)(\varphi - s\varphi'), \tag{1.1}$$

$$G_\alpha^i = \bar{G}_\alpha^i + \theta y^i - \sigma(k_1\alpha^2 + k_2\beta^2)b^i, \tag{1.2}$$

$$b_{ij} = 2\sigma[(1 + k_1b^2)a_{ij} + (k_2b^2 + k_3)b_ib_j], \tag{1.3}$$

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i), \tag{1.4}$$

where  $\sigma = \sigma(x)$  is a scalar function and  $\theta$  is 1-form,  $k_1, k_2, k_3$  are constants. In this case, both  $F =$

$\alpha\varphi\left(\frac{\beta}{\alpha}\right)$  and  $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$  are Douglas metrics.

The purpose of this paper is to study the projective relation of two subclasses of  $(\alpha, \beta)$ -metric. The main results of the paper are as follows.

**Theorem 1.1.** Let  $F = \alpha + \beta - \frac{\beta^2}{\alpha}$  be an  $(\alpha, \beta)$ -metric and  $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$  be a Kropina metric on an  $n$ -dimensional manifold  $M(n \geq 3)$  where  $\alpha$  and  $\bar{\alpha}$  are two Riemannian metrics,  $\beta$  and  $\bar{\beta}$  are two non-zero 1-forms. Then  $F$  is projectively equivalent to  $\bar{F}$  if and only if they are Douglas metrics and the geodesic co-efficient of  $\alpha$  and  $\bar{\alpha}$  have the following relations

$$G_{\alpha}^i - 2\alpha^2 \tau b^i = \bar{G}_{\bar{\alpha}}^i + \frac{1}{2\bar{b}^2} (\bar{\alpha}^2 \bar{s}^i + \bar{r}_{00} \bar{b}^i) + \theta y^i, \tag{1.5}$$

Where  $b^i = \alpha^{ij} b_j$ ,  $\bar{b}^i = \bar{\alpha}^{ij} \bar{b}_j$ ,  $\bar{b}^2 = \|\bar{\beta}\|_{\bar{\alpha}}^2$  and  $\tau = \tau(x)$  is a scalar function and  $\theta = \theta_i y^i$  is a 1-form on  $M$

By [8] and [9], we obtain immediately from theorem (1.1), that

**Proposition 1.** Let  $F = \alpha + \beta - \frac{\beta^2}{\alpha}$  an  $(\alpha, \beta)$ -metric and  $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$  be a Kropina metric on a  $n$ -dimensional manifold  $M(n \geq 3)$  where  $\alpha$  and  $\bar{\alpha}$  are two Riemannian metrics,  $\beta$  and  $\bar{\beta}$  are two nonzero collinear 1-forms. Then  $F$  is projectively equivalent to  $\bar{F}$  if and only if the following equations hold:

$$G_{\alpha}^i - 2\alpha^2 \tau b^i = \bar{G}_{\bar{\alpha}}^i + \frac{1}{2\bar{b}^2} (\bar{\alpha}^2 \bar{s}^i + \bar{r}_{00} \bar{b}^i) + \theta y^i, \tag{1.6}$$

$$b_{ij} = 2\tau\{(1 - 2b^2)a_{ij} + 3b_i b_j\}, \tag{1.7}$$

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2} (\bar{b}_i \bar{s}_j - \bar{b}_j \bar{s}_i), \tag{1.8}$$

where  $b_{ij}$  denote the coefficient of the covariant derivatives of  $\beta$  with respect to  $\alpha$ .

## II. PRELIMINARIES

We say that a Finsler metric is projectively related to another Finsler metric if they have the same geodesic as point sets. In Riemannian geometry, two Riemannian metrics  $\alpha$  and  $\bar{\alpha}$  are projectively related if and only if their spray coefficients have the relation [2],

$$G_{\alpha}^i = \bar{G}_{\bar{\alpha}}^i + \lambda_{x^k} y^k y^i, \tag{2.1}$$

where  $\lambda = \lambda(x)$  is a scalar function on the based manifold and  $(x^i, y^i)$  denotes the local coordinates in the tangent bundle  $TM$ .

Two Finsler metrics  $F$  and  $\bar{F}$  on a manifold  $M$  are called projectively related if and only if their spray coefficients have the relation [2],

$$G^i = \bar{G}^i + P(y)y^i \tag{2.2}$$

where  $P(y)$  is a scalar function on  $TM \setminus \{0\}$  and homogeneous of degree one in  $y$ .

For a given Finsler metric  $L = L(x, y)$ , the geodesic of  $L$  satisfy the following ODE:

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

Where  $G^i = G^i(x, y)$  is called the geodesic coefficient, which is given by

$$G^i = \frac{1}{4} g^{il} \{ [F^2]_{x^m y^l y^m} - [F^2]_{x^l} \}.$$

Let  $\varphi = \varphi(s)$ ,  $|s| < b_0$ , be a positive  $C^\infty$  function satisfying the following

$$\varphi(s) - s\varphi'(s) + (b^2 - s^2)\varphi''(s) > 0, \quad (|s| \leq b < b_0). \tag{2.3}$$

If  $\alpha = \sqrt{a_{ij} y^i y^j}$  is a Riemannian metric and  $\beta = b_i y^i$  is 1-form satisfying  $\|\beta_x\|_{\alpha} < b_0 \forall x \in M$ , then  $F = \alpha\varphi(s)$ ,  $s = \frac{\beta}{\alpha}$ , is called an (regular)  $(\alpha, \beta)$ -metric. In this case, the fundamental form of the metric tensor induced by  $F$  is positive definite.

Let  $\nabla\beta = b_{ij} dx^i \otimes dx^j$  be covariant derivative of  $\beta$  with respect to  $\alpha$ .

Denote

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}); s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}).$$

Note that  $\beta$  is closed if and only if  $s_{ij} = 0$  [17].

Let  $s_j = b^i s_{ij}$ ,  $s_j^i = a^{il} s_{lj}$ ,  $s_0 = s_i y^i$ ,  $s_0^i = s_j^i y^j$  and  $r_{00} = r_{ij} y^i y^j$ .

The relation between the geodesic coefficients  $G^i$  of  $F$  and geodesic coefficients  $G_\alpha^i$  of  $\alpha$  is given by

$$G^i = G_\alpha^i + \alpha Q s_0^i \{-2Q \alpha s_0 + r_{00}\} + \Psi b^i + \theta \alpha^{-1} y^i, \tag{2.4}$$

Where

$$\theta = \frac{\varphi \varphi' - s(\varphi \varphi'' + \varphi' \varphi')}{2\varphi\{(\varphi - s\varphi') + (b^2 - s^2)\varphi''\}}$$

$$Q = \frac{\varphi'}{\varphi - s\varphi'}$$

$$\Psi = \frac{1}{2} \frac{\varphi''}{\{(\varphi - s\varphi') + (b^2 - s^2)\varphi''\}}$$

For a Kropina metric  $F = \frac{\alpha^2}{\beta}$ , it is very easy to see that it is not a regular  $(\alpha, \beta)$ -metric but the relation  $\varphi(s) - s\varphi'(s) + (b^2 - s^2)\varphi''(s) > 0$  is still true for  $|s| > 0$ .

In [8], the authors characterized the  $(\alpha, \beta)$ -metrics of Douglas type.

**Lemma 2.2.** [8]: Let  $F = \alpha\varphi\left(\frac{\beta}{\alpha}\right)$  be a regular  $(\alpha, \beta)$ -metric on an  $n$ -dimensional manifold  $M(n \geq 3)$ . Assume that  $\beta$  is not parallel with respect to  $\alpha$  and  $db \neq 0$  everywhere or  $b = \text{constant}$  and  $F$  is not of Randers type. Then  $F$  is a Douglas metric if and only if the function  $\varphi = \varphi(s)$  with  $\varphi(0) = 1$  satisfies the following ODE's

$$[1 + (k_1 + k_2 s^2)s^2 + k_3 s^2]\varphi'' = (k_1 + k_2 s^2)(\varphi - s\varphi'), \tag{2.5}$$

and  $\beta$  satisfies

$$b_{i|j} = 2\sigma[(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j] \tag{2.6}$$

Where  $b^2 = \|\beta\|_\alpha^2$  and  $\sigma = \sigma(x)$  is a scalar function and  $k_1, k_2, k_3$  are constants  $(k_2, k_3) \neq (0, 0)$ .

For a Kropina metric, we have the following,

**Lemma 2.3.**[9]: Let  $F = \frac{\alpha^2}{\beta}$  be Kropina metric on an  $n$ -dimensional manifold  $M$ . Then

(i)  $(n \geq 3)$  Kropina metric  $F$  with  $b^2 \neq 0$  is Douglas metric if and only if

$$s_{ik} = \frac{1}{b^2}(b_i s_k - b_j s_i). \tag{2.7}$$

(ii)  $(n = 2)$  Kropina metric  $F$  is a Douglas metric.

**Definition 2.1.** [2]: Let

$$D_{jkl}^i = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left( G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right) \tag{2.8}$$

Where  $G^i$  is the spray coefficients of  $F$ . The tensor  $D = D_{jkl}^i \partial_i \otimes dx^j \otimes dx^k \otimes dx^l$  is called the Douglas tensor.

A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

We know that the Douglas tensor is a projective invariant [12]. Note that the spray coefficients of a Riemannian metric are quadratic forms and one can see that the Douglas tensor vanishes from (2.8). This shows that Douglas tensor is a non-Riemannian quantity.

In the following, we use quantities with a bar to denote the corresponding quantities of the metric  $\bar{F}$ .

Now, first we compute the Douglas tensor of a general  $(\alpha, \beta)$ -metric.

Let

$$\hat{G}^i = G_\alpha^i + \alpha Q s_0^i + \Psi\{-2Q \alpha s_0 + r_{00}\} b^i, \tag{2.9}$$

then (2.4) becomes

$$G^i = \hat{G}^i + \theta\{-2Q\alpha s_0 + r_{00}\}\alpha^{-1}y^i.$$

Clearly,  $G^i$  and  $\hat{G}^i$  are projective equivalent according to (2.2), they have the same Douglas tensor. Let

$$T^i = \alpha Qs_0^i + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i. \tag{2.10}$$

Then  $\hat{G}^i = G^i + T^i$ , thus

$$\begin{aligned} D_{jkl}^i &= \hat{D}_{jkl}^i, \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left( G^i - \frac{1}{n+1} \frac{\partial G_\alpha^m}{\partial y^m} y^i + T^i - \frac{1}{n+1} \frac{\partial T^m}{\partial y^m} y^i \right) \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left( T^i - \frac{1}{n+1} \frac{\partial T^m}{\partial y^m} y^i \right) \end{aligned} \tag{2.11}$$

To compute (2.11) explicitly, we use the following identities

$$\alpha_{y^k} = \alpha^{-1}y_k, \quad s_{y^k} = \alpha^{-2}(b_k\alpha - sy_k),$$

where  $y_i = a_{il}y^l$ . Here after,  $\alpha_{y^k}$  means  $\frac{\partial \alpha}{\partial y^k}$ . Then

$$[\alpha Qs_0^m]_{y^m} = \alpha^{-1}y_m Qs_0^m + \alpha^{-2}Q'[b_m\alpha^2 - \beta y_m]s_0^m = Q's_0,$$

and

$$[\Psi(-2Q\alpha s_0 + r_{00})b^m]_{y^m} = \Psi'\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] + 2\Psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0]$$

where  $r_i = b^i r_{ij}$  and  $r_0 = r_i y^i$ . Thus from (2.10), we have

$$T_{y^m}^m = Q's_0 + \Psi'\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] + 2\Psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0]. \tag{2.12}$$

Let  $F$  and  $\bar{F}$  be two  $(\alpha, \beta)$ -metrics, we assume that they have the same Douglas tensor, i.e.

$$D_{jkl}^i = \bar{D}_{jkl}^i.$$

From (2.8) and (2.11), we have

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left( T^i - \bar{T}^i - \frac{1}{n+1} (T_{y^m}^m - \bar{T}_{y^m}^m) y^i \right) = 0$$

Then there exists a class of scalar function  $H_{jk}^i = H_{jk}^i(x)$ , such that

$$H_{00}^i = T^i - \bar{T}^i - \frac{1}{n+1} (T_{y^m}^m - \bar{T}_{y^m}^m) y^i, \tag{2.13}$$

where  $H_{00}^i = H_{jk}^i y^j y^k$ ,  $T^i$  and  $T_{y^m}^m$  are given by (2.10) and (2.12) respectively

### III. PROJECTIVE RELATION OF CLASSES OF $(\alpha, \beta)$ -METRICS

In this section, we find the projective relation between special metric  $(\alpha, \beta)$ -metric  $F = \alpha + \beta - \frac{\beta^2}{\alpha}$  and  $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$  on a same underlying manifold  $M$  of dimension  $n \geq 3$ .

For  $(\alpha, \beta)$ -metric  $F = \alpha + \beta - \frac{\beta^2}{\alpha}$ , one can prove by (2.3) that  $F$  is a regular Finsler metric if and only if 1-form  $\beta$  satisfies the condition  $\|\beta_x\|_\alpha < 1$  for any  $x \in M$ .

The geodesic coefficients are given by (2.4) with

$$\begin{aligned} \theta &= \frac{\{1 + 3s^2 - 4s^3\}}{2\{1 + s - s^2\}\{1 - 2b^2 + 3s^2\}}, \\ Q &= \frac{1 - 2s}{1 + s^2}, \\ \Psi &= -\frac{1}{1 - 2b^2 + 3s^2}, \end{aligned} \tag{3.1}$$

For Kropina metric  $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ , the geodesic coefficient are given by (2.4) with

$$\begin{aligned} \bar{Q} &= -\frac{1}{2s} \\ \bar{\theta} &= -\frac{s}{\bar{b}^2} \\ \bar{\Psi} &= \frac{1}{2\bar{b}^2}. \end{aligned} \tag{3.2}$$

In this paper we assume that  $\lambda = \frac{1}{n+1}$ . Since the Douglas tensor is a projective invariant,

we have,

**Theorem 3.2.** Let  $F = \alpha + \beta - \frac{\beta^2}{\alpha}$  be an  $(\alpha, \beta)$ - metric and  $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$  be a Kropina metric on an n-dimensional manifold  $M(n \geq 3)$  where  $\alpha$  and  $\bar{\alpha}$  are two Riemannian metrics,  $\beta$  and  $\bar{\beta}$  are two non zero 1-forms. Then  $F$  and  $\bar{F}$  have the same Douglas tensors if and only if they are all Douglas metrics.

**Proof:** First, we prove the sufficient condition.

Let  $F$  and  $\bar{F}$  be Douglas metrics and corresponding Douglas tensors be  $D_{jkl}^i$  and  $\bar{D}_{jkl}^i$ . Then by the definition of Douglas metric, we have  $D_{jkl}^i = 0$  and  $\bar{D}_{jkl}^i = 0$ , that is both  $F$  and  $\bar{F}$  have the same Douglas tensor, then (2.13) holds.

Plugging (3.1) and (3.2) into (2.13), we have

$$H_{00}^i = \frac{A^i \alpha^9 + B^i \alpha^8 + C^i \alpha^7 + D^i \alpha^6 + E^i \alpha^5 + F^i \alpha^4 + G^i \alpha^3 + H^i \alpha^2 + I^i}{J \alpha^8 + K \alpha^6 + L \alpha^4 + M \alpha^2 + N} + \frac{\bar{A}^i \bar{\alpha}^2 + \bar{B}^i}{2 \bar{b}^2 \bar{\beta}} \quad (3.3)$$

where

$$\begin{aligned} A^i &= (1 - 2b^2)\{s_0^i + 2s_0 b^i - 2b^2 s_0^i\}, \\ B^i &= (1 - 2b^2)\{4b^2 \beta s_0^i - 4\beta s_0 b^i - r_{00} b^i + 2\lambda y^i (r_0 + s_0) - 2\beta s_0^i\}, \\ C^i &= \beta[\beta\{(4b^2(b^2 - 4) + 7)s_0^i + 4(2 - b^2)s_0 b^i\} + 4(1 + b^2)\lambda s_0 y^i], \\ D^i &= \beta[-2\beta^3\{(4b^2(b^2 - 4) + 7)s_0^i + (8 - 4b^2)s_0 b^i\} + (1 + b^2)\lambda s_0 b^i - \beta r_{00} b^i (4b^2 - 5) - 2\lambda y^i\{3\beta^2 r_{00} + \beta((4b^2 - 5)r_0 + (12b^2 - 3)s_0)\}], \\ E^i &= \beta^3[3\beta\{5s_0^i + 2s_0 b^i - 4b^2 s_0^i\} + (4 - 4b^2)s_0 \lambda y^i], \\ F^i &= \beta^3[6\beta^2\{4b^2 s_0^i - 12s_0 b^i - 5s_0^i\} - (7 - 2b^2)\beta r_{00} b^i + \{6(1 - 2b^2)r_{00} + \beta((14 - 4b^2)r_0 + (6 - 12b^2)s_0)\}\lambda y^i], \\ G^i &= 9\beta^6 s_0^i, \\ H^i &= -3\beta^5[\beta\{6\beta s_0^i + b^i r_{00}\} + 6\lambda y^i\{(b^2 - 2)r_{00} - \beta(r_0 + s_0)\}], \\ I^i &= 6\beta^7 r_{00} \lambda y^i \end{aligned}$$

And

$$\begin{aligned} J &= (1 - 2b^2)^2, \\ K &= 4\beta^2(1 - 2b^2)(2 - b^2), \\ L &= 2\beta^4(11 + 2b^4 - 14b^2), \\ M &= -12\beta^6(b^2 - 2), \\ N &= 9\beta^8 \end{aligned}$$

And

$$\begin{aligned} \bar{A}^i &= \bar{b}^2 \bar{s}_0^i - \bar{b}^i \bar{s}_0, \\ \bar{B}^i &= \bar{\beta}[2\lambda y^i(\bar{r}_0 + \bar{s}_0) - \bar{b}^i \bar{r}_{00}]. \end{aligned}$$

Further, (3.3) is equivalent to

$$(A^i \alpha^9 + B^i \alpha^8 + C^i \alpha^7 + D^i \alpha^6 + E^i \alpha^5 + F^i \alpha^4 + G^i \alpha^3 + H^i \alpha^2 + I^i)(2\bar{b}^2 \bar{\beta}) + (\bar{A}^i \bar{\alpha}^2 + \bar{B}^i) \times (J \alpha^8 + K \alpha^6 + L \alpha^4 + M \alpha^2 + N) = H_{00}^i (2\bar{b}^2 \bar{\beta})(J \alpha^8 + K \alpha^6 + L \alpha^4 + M \alpha^2 + N) \quad (3.4)$$

Replacing  $(y^i)$  by  $(-y^i)$  in (3.4) yields

$$(-A^i \alpha^9 + B^i \alpha^8 - C^i \alpha^7 + D^i \alpha^6 - E^i \alpha^5 + F^i \alpha^4 - G^i \alpha^3 + H^i \alpha^2 + I^i)(-2\bar{b}^2 \bar{\beta}) - (\bar{A}^i \bar{\alpha}^2 + \bar{B}^i) \times (J \alpha^8 + K \alpha^6 + L \alpha^4 + M \alpha^2 + N) = -H_{00}^i (J \alpha^8 + K \alpha^6 + L \alpha^4 + M \alpha^2 + N)(2\bar{b}^2 \bar{\beta}) \quad (3.5)$$

Adding (3.4) and (3.5), we get

$$(A^i \alpha^9 + C^i \alpha^7 + E^i \alpha^5 + G^i \alpha^3)(2\bar{b}^2 \bar{\beta}) = 0$$

Above equation reduces to

$$A^i \alpha^9 + C^i \alpha^7 + E^i \alpha^5 + G^i \alpha^3 = 0 \quad (3.6)$$

Therefore, we conclude that (3.3) is equivalent to

$$H_{00}^i = \frac{B^i\alpha^8 + D^i\alpha^6 + F^i\alpha^4 + H^i\alpha^2 + I^i}{J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N} + \frac{\bar{A}^i\bar{\alpha}^2 + \bar{B}^i}{2\bar{b}^2\bar{\beta}} \tag{3.7}$$

(3.7) is equivalent to

$$(J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N) = H_{00}^i(2\bar{b}^2\bar{\beta})(J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N) + (\bar{A}^i\bar{\alpha}^2 + \bar{B}^i) \times \tag{3.8}$$

In the above equation (3.8), we can see that  $\bar{A}^i\bar{\alpha}^2(J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N)$  can be divided by  $\bar{\beta}$ . Since  $\beta = \mu\bar{\beta}$ , then  $\bar{A}^i\bar{\alpha}^2J\alpha^8$  can be divided by  $\bar{\beta}$ . Because  $\bar{\beta}$  is prime with respect to  $\alpha$  and  $\bar{\alpha}$ . Therefore  $\bar{A}^i = \bar{b}^2\bar{s}_0^i - \bar{b}^i\bar{s}_0$  can be divided by  $\bar{\beta}$ . Hence there is a scalar function  $\Psi^i(x)$  such that

$$\bar{b}^2\bar{s}_0^i - \bar{b}^i\bar{s}_0 = \bar{\beta}\Psi^i \tag{3.9}$$

Transvecting (3.9) by  $\bar{y}_i = \bar{a}_{ij}y^j$ , we get  $\Psi^i(x) = -\bar{s}^i$ . Thus we have

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i) \tag{3.10}$$

Thus, by lemma 2.3,  $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$  is a Douglas metrics. i.e. Both  $F = \alpha + \beta - \frac{\beta^2}{\alpha}$ , and  $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$  are Douglas metrics.

If  $n = 2$ ,  $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$  is a Douglas metric by lemma 2.3. Thus  $F$  and  $\bar{F}$  have the same Douglas tensors means that they are Douglas metrics. Thus  $F$  and  $\bar{F}$  have the same Douglas tensors means that they are Douglas metrics. Thus  $F = \alpha + \beta - \frac{\beta^2}{\alpha}$  be an special  $(\alpha, \beta)$  –metric and  $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$  be a Kropina metric on an  $n$ -dimensional manifold  $M(n \geq 2)$ , where  $\alpha$  and  $\bar{\alpha}$  are Riemannian metric,  $\beta$  and  $\bar{\beta}$  are two non zero collinear 1-forms. Then  $F$  and  $\bar{F}$  have same Douglas tensors if and only if they are Douglas metrics. This completes the proof of theorem (3.2).

#### IV. PROOF OF THEOREM 1.1.

First, we prove the necessary condition:

Since Douglas tensor is an invariant under projective changes between two Finsler metrics, If  $F$  is projectively related to  $\bar{F}$ , then they have the same Douglas tensor. According to theorem (3.2), we obtain that both  $F$  and  $\bar{F}$  are Douglas metrics.

By [3], It is well known that Kropina metric  $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$  with  $b^2 \neq 0$  is a Douglas metric if and only if  $s_{ik} = \frac{1}{b^2}(b_i s_k - b_k s_i)$  and also it has been proved that by [7], we know that  $(\alpha, \beta)$  –metric,  $F = \alpha + \beta - \frac{\beta^2}{\alpha}$  is a Douglas metric if and only if

$$b_{i|j} = 2\tau\{(1 - 2b^2)a_{ij} + 3b_i b_j\} \tag{4.1}$$

where  $\tau = \tau(x)$  is a scalar function on  $M$ . In this case,  $\beta$  is closed.

Plugging (4.1) and (3.1) into (2.4), we have

$$G^i = G_\alpha^i + \left(\frac{\alpha^3 + 3\alpha\beta^2 - 4\beta^3}{\alpha^2 + \alpha\beta - \beta^2}\right)\tau y^i - 2\tau\alpha^2 b^i \tag{4.2}$$

Again plugging (3.10) and (3.2) into (2.4), we have

$$\bar{G}^i = \bar{G}_\alpha^i + \frac{1}{2\bar{b}^2} \left\{ -\bar{\alpha}^2 \bar{s}^i + (2\bar{s}_0 y^i - \bar{r}_{00} \bar{b}^i) + 2 \frac{\bar{r}_{00} \bar{\beta} y^i}{\bar{\alpha}^2} \right\} \tag{4.3}$$

Since F is Projectively equivalent to  $\bar{F}$ , then their exist a scalar function  $P = P(x, y)$  on  $TM \setminus \{0\}$  such that

$$G^i = \bar{G}^i + Py^i \tag{4.4}$$

By (4.2), (4.3) and (4.4), we have

$$\left[ P - \left( \frac{\alpha^3 + 3\alpha\beta^2 - 4\beta^3}{\alpha^2 + \alpha\beta - \beta^2} \right) \tau - \frac{1}{\bar{b}^2} \left( \bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2} \right) \right] y^i = G_\alpha^i - \bar{G}_\alpha^i - 2\alpha^2 \tau b^i - \frac{1}{2\bar{b}^2} (\bar{\alpha}^2 \bar{s}^i + \bar{r}_{00} \bar{b}^i) \tag{4.5}$$

Note that RHS of above equation is in quadratic form.

Then there must be a one form  $\theta = \theta_i y^i$  on M, such that

$$\left[ P - \left( \frac{\alpha^3 + 3\alpha\beta^2 - 4\beta^3}{\alpha^2 + \alpha\beta - \beta^2} \right) \tau - \frac{1}{\bar{b}^2} \left( \bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2} \right) \right] = \theta$$

Thus (4.5) becomes

$$G_\alpha^i - 2\alpha^2 \tau b^i = \bar{G}_\alpha^i + \frac{1}{2\bar{b}^2} (\bar{\alpha}^2 \bar{s}^i + \bar{r}_{00} \bar{b}^i) + \theta y^i \tag{4.6}$$

This completes the proof of necessity.

Conversely from (4.2), (4.3) and (1.5) we have

$$G^i = \bar{G}^i + \left[ \theta + \left( \frac{\alpha^3 + 3\alpha\beta^2 - 4\beta^3}{\alpha^2 + \alpha\beta - \beta^2} \right) \tau + \frac{1}{\bar{b}^2} \left( \bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2} \right) \right] y^i \tag{4.7}$$

Thus F is projectively equivalent to  $\bar{F}$ . From the above theorem, immediately we get the following corollary

**Corollary 4.1.** [18]: Let  $L = \alpha + \beta - \frac{\beta^2}{\alpha}$  be a special  $(\alpha, \beta)$ -metric and  $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$  be a Kropina metric be two  $(\alpha, \beta)$ -metrics on a n-dimensional manifold M with dimension  $n \geq 3$ , where  $\alpha$  and  $\bar{\alpha}$  are two Riemannian metrics,  $\beta$  and  $\bar{\beta}$  are two non-zero collinear 1-forms. Then F is projectively related to  $\bar{F}$  if and only if they are Douglas metrics and the spray coefficients of  $\alpha$  and  $\bar{\alpha}$  have the following relations

$$\begin{aligned} G^i - 2\alpha^2 \tau b^i &= \bar{G}_\alpha^i + \frac{1}{2\bar{b}^2} (\bar{\alpha}^2 \bar{s}^i + \bar{r}_{00} \bar{b}^i) + \theta y^i, \\ s_{ij} &= 0 \\ \bar{s}_{ij} &= \frac{1}{\bar{b}^2} (\bar{b}_i \bar{s}_j - \bar{b}_j \bar{s}_i) \\ b_{ij} &= 2\tau \{ (1 - 2b^2) a_{ij} + 3b_i b_j \} \end{aligned}$$

Where  $b_{ij}$  denotes the coefficients of the covariant derivative of  $\beta$  with respect to  $\alpha$ .

**REFERENCES**

1. S. Bacso and M. Matsumot, Projective change between FInslers space with  $(\alpha, \beta)$ - metric, Tensor N.S. 55 (1994), 252-257.
2. N. Cui and Yi-Bing, Projective change between two classes of  $(\alpha, \beta)$ -metrics, Diff.Geom. and its Applications 27 (2009), 566-573.
3. Feng Mu and Xinyue Cheng, On the Projective Equivalence between  $(\alpha, \beta)$ -metrics and Kropina metric, Diff. Geom-Dynamical systems, Vol.14, (2012), 106-116.
4. Z. M. Haasiguchi and Y. Ichijyo, Randers space with rectilinear geodesics, Rep. Fac.Sci.Kagoshima.Uni, (Math. Phys.Chen), 13, (1980) 33-40.
5. R. S. Ingarden, Geometry of thermodynamics, Diff. Geom. Methods in Theor. Phys, XV Intern. Conf.Clausthal 1986, World Scientific, Singapore, 1987.
6. Jiang Jingnang and Cheng Xinyue, Projective change between two Important classes of  $(\alpha, \beta)$ -metrics, Advances in Mathematics, Vol.06, (2012).
7. V. K. Kropina, On the Projective Finsler space with certain special form, Nauchn. Doklady vyss. Skoly, Fiz-mat. Nauki, 1952(2)(1960), 38-42 (Russian).14
8. B. Li, Y. Shen and Z. Shen, On a Class of Douglas metrics, Studia Scientiarum Mathematicarum Hungarica, 46(3) (2009), 355-365.
9. M. Matsumto, Finsler Space with  $(\alpha, \beta)$ -metric of douglas type, Tensor N.S. 60 (1998).
10. M. Matsumto and S. i. Hojo, A Conclusive theorem on C-reducible Finsler spaces, Tensors, N.S, 32 (1978), 225-230.
11. S. K. Narasimhamurthy, Projective change between Matsumoto metric and Randers metric, Proc. Jangjeon Math. Soc, No.03, 393-402 (2014).
12. H. S. park and Il-Yong Lee, Randers change of Finsler space with  $(\alpha, \beta)$ -metric of Douglas Type, J.Korean Math. Soc.38 (3) (2001), 503-521.

13. Pradeep Kumar, Madhu T S and Ramesha M, Projective equivalence between two Families of Finsler metrics, *Gulf Journal of Mathematics*, 4(1)(2016), 65-74.
14. Pradeep Kumar, Ramesha M and Madhu T S, On two important classes of  $(\alpha, \beta)$ -metrics being projectively related, *International Journal of Current Research*, 10(6)(2018), 70528-70536.
15. C. Shibata, On a Finsler space with  $(\alpha, \beta)$ -metric, *J. Hokkaido Uni. of Education*, IIA 35 (1984), 1-6.
16. P. Stavrinou, F. Diakogiannis, Finslerian structure of anisotropic gravitational field, *Gravit. Cosmol.*, 10 (4) (2004), 1-11.
17. Z. Shen, On a Landsberg  $(\alpha, \beta)$ -metric, (2006).
18. A. Tayebi, Sadeghi and E. Peyghan, Two Families of Finsler metrics Projectively related to a Kropina metric, *arxiv:1302.4435v1[math.Dg]*, (2013).
19. A. Tayebi, E. Peyghan and H. Sadeghi, On two subclasses of  $(\alpha, \beta)$ -metrics being projectively related, *Journal of Geometry and Physics*, 62 (2012), 292-300.
20. M.Zohrehv and M.M.Rezaei, On Projective related two special classes of  $(\alpha, \beta)$ -metrics, *Differential geometry and its applications*, 29 (2011), 660-669..