Curvature and Projective Relations of Specialized (α,β)-Metrics in Finsler Geometry

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ABSTRACT
In this paper, we find the necessary and sufficient condition to characterize the projective relation between two subclasses of (α,β)-metrics $L = \alpha + \beta - \frac{\beta^2}{\alpha}$ and $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ on a manifold M with dimension $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non-zero 1-forms.

Keywords: *Finsler space, (α,β)metric, Kropina metric, Projective change, Douglas metric.*
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I. INTRODUCTION

In Finsler geometry, two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if $G^i = \bar{G}^i + Py^i$, where G^i and $\bar{G}^i$ are the geodesic coefficients of F and $\bar{F}$ respectively and $P = P(x,y)$ is a scalar function on the slit tangent bundle TM_0. In this case, any geodesic of the first is also geodesic for the second and viceversa. The projective changes between two Finsler spaces have been studied by [1], [2], [3], [4], [6], [11],[13],[14], [18], [19], [20].

(α,β)-metrics form a special and very important classes of Finsler metrics which can be expressed in the for $F = \alpha\varphi(s)$: $s = \frac{\beta}{\alpha}$, where α is a Riemannian metric and β is a 1-form and φ is a C^∞ positive function on the definite domain. In particular, when $\varphi = \frac{1}{s}$, the Finsler metric $F = \frac{\beta^2}{\alpha}$ is called Kropina metric. Kropina metric was first introduced by L. Berwald in connection with two dimensional Finsler space with rectilinear extremal and was investigated by V.K. Kropina [7]. They together with Randers metric are C-reducible [10]. However, Randers metric are regular Finsler metric but Kropina metric is non-regular Finsler metric. Kropina metric seem to be among the simplest nontrivial Finsler metric with many interesting applications in physics, electron optics with a magnetic field, dissipative mechanics and irreversible thermodynamics [5], [15]. Also, there are interesting applications in relativistic field theory, evolution and developmental biology.

Based on Stavrino's work on Finslerian structure of anisotropic gravitational field [16], we know that the anisotropy is an issue of the background radiation for all possible (α,β)-metrics. Then the 1-form β represents the same direction of the observed anisotropy of the microwave background radiation. That is, if two (α,β)-metrics $F = \alpha\varphi\left(\frac{\beta}{\alpha}\right)$ and $\bar{F} = \bar{\alpha}\varphi\left(\frac{\bar{\beta}}{\bar{\alpha}}\right)$ are the same anisotropy directions (or, they have the same axis rotation to their indicatrices), then their 1-form.

β and $\bar{\beta}$ are collinear, there is a function $\mu \epsilon C^\infty(M)$ such that $\beta(x,y) = \mu\bar{\beta}(x,y)$. By [3], for the projective equivalence between a general (α,β)-metric and a Kropina metric, we have the following lemma:

Lemma 1.1. Let $F = \alpha\varphi\left(\frac{\beta}{\alpha}\right)$be an (α,β)-metricon n-dimensional manifold $M(n \geq 3)$, satisfying that β is not parallel with respect to $\alpha, db \neq 0$ everywhere (or) b = constant and F is not of Randers type. Let $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on the manifold M, where $\bar{\alpha} = \lambda(x)\alpha$ and $\bar{\beta} = \mu(x)\beta$. Then F is Projectively Equivalent to $\bar{F}$ if and only if the following equations holds,

$$[1 + (k_1 + k_2 s^2)s^2 + k_3 s^2]\varphi'' = (k_1 + k_2 s^2)(\varphi - s\varphi'), \quad (1.1)$$

$$G^i_\alpha = \bar{G}^i_{\bar{\alpha}} + \theta y^i - \sigma(k_1\alpha^2 + k_2\beta^2)b^i, \quad (1.2)$$

$$b_{i|j} = 2\sigma\left[(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j\right], \quad (1.3)$$

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}\left(\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i\right), \quad (1.4)$$

where $\sigma = \sigma(x)$ is a scalar function and θ is 1-form, k_1, k_2 k_3 are constants. In this case, both $F =$

$\alpha\varphi\left(\frac{\beta}{\alpha}\right)$ and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ are Douglas metrics.

The purpose of this paper is to study the projective relation of two subclasses of (α, β)-metric. The main results of the paper are as follows.

Theorem 1.1. Let $F = \alpha + \beta - \frac{\beta^2}{\alpha}$ be an (α, β)-metric and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on an n-dimensional manifold $M(n \geq 3)$ where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non-zero 1-forms. Then F is projectively equivalent to $\bar{F}$ if and only if they are Douglas metrics and the geodesic co-efficient of α and $\bar{\alpha}$ have the following relations

$$G^i_\alpha - 2\alpha^2 \tau b^i = \bar{G}^i_{\bar{\alpha}} + \frac{1}{2\bar{b}^2}\left(\bar{\alpha}^2 \bar{s}^i + \bar{r}_{00}\bar{b}^i\right) + \theta y^i, \quad (1.5)$$

Where $b^i = a^{ij}b_j,\ \bar{b}^i = \bar{a}^{ij}\bar{b}_j,\ \bar{b}^2 = \|\bar{\beta}\|^2_{\bar{\alpha}}$ and $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on M

By [8] and [9], we obtain immediately from theorem (1.1), that

Proposition 1. Let $F = \alpha + \beta - \frac{\beta^2}{\alpha}$ an (α, β)-metric and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on a n-dimensional manifold $M(n \geq 3)$ where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero collinear 1-forms. Then F is projectively equivalent to $\bar{F}$ if and only if the following equations hold:

$$G^i_\alpha - 2\alpha^2 \tau b^i = \bar{G}^i_{\bar{\alpha}} + \frac{1}{2\bar{b}^2}\left(\bar{\alpha}^2 \bar{s}^i + \bar{r}_{00}\bar{b}^i\right) + \theta y^i, \quad (1.6)$$

$$b_{i|j} = 2\tau\{(1 - 2b^2)a_{ij} + 3b_i b_j\}, \quad (1.7)$$

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}\left(\bar{b}_i \bar{s}_j - \bar{b}_j \bar{s}_i\right), \quad (1.8)$$

where $b_{i|j}$ denote the coefficient of the covariant derivatives of β with respect to α.

II. PRELIMINARIES

We say that a Finsler metric is projectively related to another Finsler metric if they have the same geodesic as point sets. In Riemannian geometry, two Riemannian metrics α and $\bar{\alpha}$ are projectively related if and only if their spray coefficients have the relation [2],

$$G^i_\alpha = G^i_{\bar{\alpha}} + \lambda_{x^k} y^k y^i, \quad (2.1)$$

where $\lambda = \lambda(x)$ is a scalar function on the based manifold and (x^i, y^i) denotes the local coordinates in the tangent bundle TM.

Two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if and only if their spray coefficients have the relation [2],

$$G^i = \bar{G}^i + P(y)y^i \quad (2.2)$$

where $P(y)$ is a scalar function on $TM\backslash\{0\}$ and homogeneous of degree one in y.
For a given Finsler metric $L = L(x, y)$, the geodesic of L satisfy the following ODE:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

Where $G^i = G^i(x, y)$ is called the geodesic coefficient, which is given by

$$G^i = \frac{1}{4}g^{il}\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\}.$$

Let $\varphi = \varphi(s),\ |s| < b_0,$ $be\ a\ positive\ C^\infty$ function satisfying the following

$$\varphi(s) - s\varphi'(s) + (b^2 - s^2)\varphi''(s) > 0, \quad \left(|s| \leq b < b_0\right). \quad (2.3)$$

If $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is 1-form satisfying $\|\beta_x\|_\alpha < b_0 \forall x \in M, then\ F = \alpha\varphi(s)$, $s = \frac{\beta}{\alpha}$, is called an (regular) (α, β)-metric. In this case, the fundamental form of the metric tensor induced by F is positive definite.

Let $\nabla\beta = b_{i|j}dx^i \otimes dx^j$ be covariant derivative of β with respect to α.

Denote

$$r_{ij} = \frac{1}{2}\left(b_{i|j} + b_{j|i}\right);\ s_{ij} = \frac{1}{2}\left(b_{i|j} - b_{j|i}\right).$$

Note that β is closed if and only if $s_{ij} = 0$ [17].

Let $s_j = b^i s_{ij},\ s_j^i = a^{il} s_{lj},\ s_0 = s_i y^i,\ s_0^i = s_j^i y^j\ and\ r_{00} = r_{ij} y^i y^j$.

The relation between the geodesic coefficients $G^i\ of\ F$ and geodesic coefficients G_α^i of α is given by

$$G^i = G_\alpha^i + \alpha Q s_0^i\{-2Q\alpha s_0 + r_{00}\} + \Psi b^i + \theta\alpha^{-1} y^i, \qquad (2.4)$$

Where

$$\theta = \frac{\varphi\varphi' - s(\varphi\varphi'' + \varphi'\varphi')}{2\varphi\{(\varphi - s\varphi') + (b^2 - s^2)\varphi''\}}$$

$$Q = \frac{\varphi'}{\varphi - S\varphi'}$$

$$\Psi = \frac{1}{2}\frac{\varphi''}{\{(\varphi - S\varphi') + (b^2 - s^2)\varphi''\}}$$

For a Kropina metric $F = \frac{\alpha^2}{\beta}$, it is very easy to see that it is not a regular (α, β)-metric but the relation $\varphi(s) - s\varphi'(s) + (b^2 - s^2)\varphi''(s) > 0$ is still true for $|s| > 0$.

In [8], the authors characterized the (α, β)-metrics of Douglas type.

Lemma 2.2. [8]: Let $F = \alpha\varphi\left(\frac{\beta}{\alpha}\right)$ be a regular (α, β)-metric on an n-dimensional manifold $M(n \geq 3)$. Assume that β is not parallel with respect to α and $db \neq 0$ everywhere or b= constant and F is not of Randers type. Then F is a Douglas metric if and only if the function $\varphi = \varphi(s)$ with $\varphi(0) = 1$ satisfies the following ODE's

$$[1 + (k_1 + k_2 s^2)s^2 + k_3 s^2]\varphi'' = (k_1 + k_2 s^2)(\varphi - s\varphi'), \qquad (2.5)$$

and β satisfies

$$b_{i|j} = 2\sigma\left[(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j\right] \qquad (2.6)$$

Where $b^2 = \|\beta\|_\alpha^2\ and\ \sigma = \sigma(x)$ is a scalar function and k_1, k_2, k_3 are constants $(k_2, k_3) \neq (0,0)$.

For a Kropina metric, we have the following,

Lemma 2.3.[9]: Let $F = \frac{\alpha^2}{\beta}$ be Kropina metric on an n-dimensional manifold M. Then

(i) $(n \geq 3)$Kropina metric F with $b^2 \neq 0$ is Douglas metric if and only if

$$s_{ik} = \frac{1}{b^2}\left(b_i s_k - b_j s_i\right). \qquad (2.7)$$

(ii) $(n = 2)$ Kropina metric F is a Douglas metric.

Definition 2.1. [2]: Let

$$D_{jkl}^i = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right) \qquad (2.8)$$

Where G^i is the spray coefficients of F. The tensor $D = D_{jkl}^i \partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called the Douglas tensor. A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

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We know that the Douglas tensor is a projective invariant [12]. Note that the spray coefficients of a Riemannian metric are quadratic forms and one can see that the Douglas tensor vanishes from (2.8). This shows that Douglas tensor is a non-Riemannian quantity.

In the following, we use quantities with a bar to denote the corresponding quantities of the metric $\bar{F}$.

Now, first we compute the Douglas tensor of a general (α, β)-metric.

Let

$$\hat{G}^i = G_\alpha^i + \alpha Q s_0^i + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i, \qquad (2.9)$$

then (2.4) becomes

$$G^i = \hat{G}^i + \theta\{-2Q\alpha s_0 + r_{00}\}\alpha^{-1}y^i.$$

Clearly, G^i *and* $\hat{G}^i$ are projective equivalent according to (2.2), they have the same Douglas tensor.
Let

$$T^i = \alpha Q s_0^i + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i. \qquad (2.10)$$

Then $\hat{G}^i = G_\alpha^i + T^i$, thus

$$D_{jkl}^i = \hat{D}_{jkl}^i,$$
$$= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G_\alpha^i - \frac{1}{n+1}\frac{\partial G_\alpha^m}{\partial y^m}y^i + T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right)$$
$$= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right) \qquad (2.11)$$

To compute (2.11) explicitly, we use the following identities

$$\alpha_{y^k} = \alpha^{-1}y_k,\ s_{y^k} = \alpha^{-2}(b_k\alpha - sy_k),$$

where $y_i = a_{il}y^l$. *Here after,* α_{y^k} *means* $\frac{\partial\alpha}{\partial y^k}$. Then

$$[\alpha Q s_0^m]_{y^m} = \alpha^{-1}y_m Q s_0^m + \alpha^{-2}Q'[b_m\alpha^2 - \beta y_m]s_0^m = Q's_0,$$

and

$$[\Psi(-2Q\alpha s_0 + r_{00})b^m]_{y^m} = \Psi'\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha S_0] + 2\Psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0]$$

where $r_i = b^i r_{ij}$ *and* $r_0 = r_i y^i$. Thus from (2.10) , we have

$$T_{y^m}^m = Q's_0 + \Psi'\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] + 2\Psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0]. \qquad (2.12)$$

Let F and $\bar{F}$ be two (α,β)-metrics, we assume that they have the same Douglas tensor, i.e.

$$D_{jkl}^i = \bar{D}_{jkl}^i.$$

From (2.8) and (2.11), we have

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \bar{T}^i - \frac{1}{n+1}\left(T_{y^m}^m - \bar{T}_{y^m}^m\right)y^i\right) = 0$$

Then there exists a class of scalar function $H_{jk}^i = H_{jk}^i(x)$, such that

$$H_{00}^i = T^i - \bar{T}^i - \frac{1}{n+1}\left(T_{y^m}^m - \bar{T}_{y^m}^m\right)y^i, \qquad (2.13)$$

where $H_{00}^i = H_{jk}^i y^j y^k$, T^i *and* $T_{y^m}^m$ are given by (2.10) *and* (2.12) respectively

III. PROJECTIVE RELATION OF CLASSES OF (α,β)-METRICS

In this section, we find the projective relation between special metric (α,β)-metric $F = \alpha + \beta - \frac{\beta^2}{\alpha}$ and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ on a same underlying manifold M of dimension $n \geq 3$.

For (α,β)-metric $F = \alpha + \beta - \frac{\beta^2}{\alpha}$, one can proveby (2.3) that F is a regular Finsler metric if and only if 1-form β satisfies the condition $\|\beta_x\|_\alpha < 1$ for any $x \in M$.

The geodesic coefficients are given by (2.4) with

$$\theta = \frac{\{1 + 3s^2 - 4s^3\}}{2\{1 + s - s^2\}\{1 - 2b^2 + 3s^2\}},$$
$$Q = \frac{1 - 2s}{1 + s^2},$$
$$\Psi = -\frac{1}{1 - 2b^2 + 3s^2}, \qquad (3.1)$$

For Kropina metric $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$, the geodesic coefficient are given by (2.4) with

$$\bar{Q} = -\frac{1}{2s}$$
$$\bar{\theta} = -\frac{s}{\bar{b}^2}$$
$$\bar{\Psi} = \frac{1}{2\bar{b}^2}. \qquad (3.2)$$

In this paper we assume that $\lambda = \frac{1}{n+1}$. Since the Douglas tensor is a projective invariant, we have,

Theorem 3.2. Let $F = \alpha + \beta - \frac{\beta^2}{\alpha}$ be an (α, β)- metric and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on an n-dimensional manifold $M(n \geq 3)$ where α *and* $\bar{\alpha}$ are two Riemannian metrics, β *and* $\bar{\beta}$ are two non zero 1-forms. Then F and $\bar{F}$ have the same Douglas tensors if and only if they are all Douglas metrics.

Proof: First, we prove the sufficient condition.
Let F and $\bar{F}$ be Douglas metrics and corresponding Douglas tensors be D^i_{jkl} *and* $\bar{D}^i_{jkl}$.Then by the definition of Douglas metric, we have $D^i_{jkl} = 0$ *and* $\bar{D}^i_{jkl} = 0$, that is both F and $\bar{F}$ have the same Douglas tensor, then (2.13) holds.
Plugging (3.1) and (3.2) into(2.13), we have

$$H^i_{00} = \frac{A^i\alpha^9 + B^i\alpha^8 + C^i\alpha^7 + D^i\alpha^6 + E^i\alpha^5 + F^i\alpha^4 + G^i\alpha^3 + H^i\alpha^2 + I^i}{J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N} + \frac{\bar{A}^i\bar{\alpha}^2 + \bar{B}^i}{2\bar{b}^2\bar{\beta}} \qquad (3.3)$$

where $A^i = (1-2b^2)\{s^i_0 + 2s_0b^i - 2b^2s^i_0\}$, $B^i = (1-2b^2)\{4b^2\beta s^i_0 - 4\beta s_0b^i - r_{00}b^i + 2\lambda y^i(r_0 + s_0) - 2\beta s^i_0\}$, $C^i = \beta[\beta\{(4b^2(b^2-4)+7)s^i_0 + 4(2-b^2s_0b^i)\} + 4(1+b^2)\lambda s_0y^i]$, $D^i = \beta[-2\beta^3\{(4b^2(b^2-4)+7)s^i_0 + (8-4b^2)s_0b^i\} + (1+b^2)\lambda s_0b^i - \beta r_{00}b^i(4b^2 - 5) - 2\lambda y^i\{3\beta^2r_{00} + \beta((4b^2-5)r_0 + (12b^2-3)s_0)\}]$, $E^i = \beta^3[3\beta\{5s^i_0 + 2s_0b^i - 4b^2s^i_0\} + (4-4b^2)s_0\lambda y^i]$,
$F^i = \beta^3[6\beta^2\{4b^2s^i_0 - 12s_0b^i - 5s^i_0\} - (7-2b^2)\beta r_{00}b^i + \{6(1-2b^2)r_{00} + \beta((14 - 4b^2)r_0 + (6-12b^2)s_0)\}\lambda y^i]$, $G^i = 9\beta^6s^i_0$,
$H^i = -3\beta^5[\beta\{6\beta s^i_0 + b^ir_{00}\} + 6\lambda y^i\{(b^2-2)r_{00} - \beta(r_0 + s_0)\}]$, $I^i = 6\beta^7r_{00}\lambda y^i$

And
$J = (1-2b^2)^2$,
$K = 4\beta^2(1-2b^2)(2-b^2)$,
$L = 2\beta^4(11 + 2b^4 - 14b^2)$,
$M = -12\beta^6(b^2-2)$,
$N = 9\beta^8$

And
$\bar{A}^i = \bar{b}^2\bar{s}^i_0 - \bar{b}^i\bar{s}_0$,
$\bar{B}^i = \bar{\beta}[2\lambda y^i(\bar{r}_0 + \bar{s}_0) - \bar{b}^i\bar{r}_{00}]$.

Further, (3.3) is equivalent to

$$(A^i\alpha^9 + B^i\alpha^8 + C^i\alpha^7 + D^i\alpha^6 + E^i\alpha^5 + F^i\alpha^4 + G^i\alpha^3 + H^i\alpha^2 + I^i)(2\bar{b}^2\bar{\beta}) + (\bar{A}^i\bar{\alpha}^2 + \bar{B}^i) \times (J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N) = H^i_{00}(2\bar{b}^2\bar{\beta})(J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N) \qquad (3.4)$$

Replacing (y^i) by $(-y^i)$ in (3.4) yields

$$(-A^i\alpha^9 + B^i\alpha^8 - C^i\alpha^7 + D^i\alpha^6 - E^i\alpha^5 + F^i\alpha^4 - G^i\alpha^3 + H^i\alpha^2 + I^i)(-2\bar{b}^2\bar{\beta}) - (\bar{A}^i\bar{\alpha}^2 + \bar{B}^i) \times (J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N) = -H^i_{00}(J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N)(2\bar{b}^2\bar{\beta}) \qquad (3.5)$$

Adding (3.4) *and* (3.5), we get

$$(A^i\alpha^9 + C^i\alpha^7 + E^i\alpha^5 + G^i\alpha^3)(2\bar{b}^2\bar{\beta}) = 0$$

Above equation reduces to

$$A^i\alpha^9 + C^i\alpha^7 + E^i\alpha^5 + G^i\alpha^3 = 0 \qquad (3.6)$$

Therefore, we conclude that (3.3) is equivalent to

$$H_{00}^{i} = \frac{B^i\alpha^8+D^i\alpha^6+F^i\alpha^4+H^i\alpha^2+I^i}{J\alpha^8+K\alpha^6+L\alpha^4+M\alpha^2+N} + \frac{\bar{A}^i\bar{\alpha}^2+\bar{B}^i}{2\bar{b}^2\bar{\beta}} \qquad (3.7)$$

(3.7) is equivalent to

$$B^i\alpha^8 + D^i\alpha^6 + F^i\alpha^4 + H^i\alpha^2 + I^i)(2\bar{b}^2\bar{\beta}) + \overline{(A}^i\bar{\alpha}^2 + \bar{B}^i) \times$$
$$(J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N) = H_{00}^{i}(2\bar{b}^2\bar{\beta})(J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N) \qquad (3.8)$$

In the above equation (3.8), we can see that $\bar{A}^i\bar{\alpha}^2(J\alpha^8 + K\alpha^6 + L\alpha^4 + M\alpha^2 + N)$ can be divided by $\bar{\beta}$. Since $\beta = \mu\bar{\beta}$, then $\bar{A}^i\bar{\alpha}^2 J\alpha^8$ can be divided by $\bar{\beta}$. Because $\bar{\beta}$ is prime with respect to $\alpha\ and\ \bar{\alpha}$. Therefore $\bar{A}^i = \bar{b}^2\bar{s}_0^i - \bar{b}^i\bar{s}_0$ can be divided by $\bar{\beta}$. Hence there is a scalar function $\Psi^i(x)$ such that

$$\bar{b}^2\bar{s}_0^i - \bar{b}^i\bar{s}_0 = \bar{\beta}\Psi^i \qquad (3.9)$$

Transvecting (3.9) by $\bar{y}_i = \bar{a}_{ij}y^j$, we get $\Psi^i(x) = -\bar{s}^i$. Thus we have

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i) \qquad (3.10)$$

Thus, by lemma 2.3, $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ is a Douglas metrics. i.e. Both $F = \alpha + \beta - \frac{\beta^2}{\alpha}$,
and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ are Douglas metrics.

If $n = 2$, $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ is a Douglas metric by lemma 2.3. Thus $F and\ \bar{F}$ have the same Douglas tensors means that they are Douglas metrics. Thus $F and\ \bar{F}$ have the same Douglas tensors means that they are Douglas metrics. Thus $F = \alpha + \beta - \frac{\beta^2}{\alpha}$ be an special (α, β) −metric and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on an n-dimensional manifold $M(n \geq 2)$, where $\alpha\ and\ \bar{\alpha}$ are Riemannian metric, $\beta\ and\ \bar{\beta}$ are two non zero collinear 1-forms. Then F and $\bar{F}$ have same Douglas tensors if and only if they are Douglas metrics. This completes the proof of theorem (3.2).

IV. PROOF. OF THEOREM 1.1.

First, we prove the necessary condition:
Since Douglas tensor is an invariant under projective changes between two Finsler metrics, If F is projectively related to $\bar{F}$, then they have the same Douglas tensor. According to theorem (3.2), we obtain that both F and $\bar{F}$ are Douglas metrics.

By [3], It is well known that Kropina metric $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ with $b^2 \neq 0$ is a Douglas metric if and only if $s_{ik} = \frac{1}{b^2}(b_i s_k - b_k s_i)$ and also it has it has been proved that by [7], we know that (α, β) −metric, $F = \alpha + \beta - \frac{\beta^2}{\alpha}$ is a Douglas metric if and only if

$$b_{i|j} = 2\tau\{(1 - 2b^2)a_{ij} + 3b_i b_j\} \qquad (4.1)$$

where $\tau = \tau(x)$ is a scalar function on M. In this case, β is closed.
Plugging (4.1) and (3.1) into (2.4), we have

$$G^i = G_\alpha^i + \left(\frac{\alpha^3+3\alpha\beta^2-4\beta^3}{\alpha^2+\alpha\beta-\beta^2}\right)\tau y^i - 2\tau\alpha^2 b^i \qquad (4.2)$$

Again plugging (3.10) and (3.2) into (2.4), we have

$$\bar{G}^i = \bar{G}_\alpha^i + \frac{1}{2\bar{b}^2}\left\{-\bar{\alpha}^2\bar{s}^i + (2\bar{s}_0 y^i - \bar{r}_{00}\bar{b}^i) + 2\frac{\bar{r}_{00}\bar{\beta}y^i}{\bar{\alpha}^2}\right\} \qquad (4.3)$$

Since F is Projectively equivalent to $\bar{F}$, then their exit a scalar function $P = P(x,y)$ on $TM\backslash\{0\}$ such that

$$G^i = \bar{G}^i + Py^i \quad (4.4)$$

By (4.2), (4.3) and (4.4), we have

$$\left[P - \left(\frac{\alpha^3 + 3\alpha\beta^2 - 4\beta^3}{\alpha^2 + \alpha\beta - \beta^2}\right)\tau - \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2}\right)\right]y^i = G^i_\alpha - \bar{G}^i_{\bar{\alpha}} - 2\alpha^2\tau b^i - \frac{1}{2\bar{b}^2}\left(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i\right) \quad (4.5)$$

Note that RHS of above equation is in quadratic form.

Then there must be a one form $\theta = \theta_i y^i$ on M, such that

$$\left[P - \left(\frac{\alpha^3 + 3\alpha\beta^2 - 4\beta^3}{\alpha^2 + \alpha\beta - \beta^2}\right)\tau - \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2}\right)\right] = \theta$$

Thus (4.5) becomes

$$G^i_\alpha - 2\alpha^2\tau b^i = \bar{G}^i_{\bar{\alpha}} + \frac{1}{2\bar{b}^2}\left(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i\right) + \theta y^i \quad (4.6)$$

This completes the proof of necessity.

Conversely from (4.2), (4.3) and (1.5) we have

$$G^i = \bar{G}^i + \left[\theta + \left(\frac{\alpha^3+3\alpha\beta^2-4\beta^3}{\alpha^2+\alpha\beta-\beta^2}\right)\tau + \frac{1}{\bar{b}^2}\left(\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2}\right)\right]y^i \quad (4.7)$$

Thus F is projectively equivalent to $\bar{F}$. From the above theorem, immediately we get the following corollary

Corollary 4.1. [18]: Let $L = \alpha + \beta - \frac{\beta^2}{\alpha}$ be a special (α,β)-metric and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric be two (α,β)-metrics on a n-dimensional manifold M with dimension $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non-zero collinear 1-forms. Then F is projectively related to $\bar{F}$ if and only if they are Douglas metrics and the spray coeffcients of α and $\bar{\alpha}$ have the following relations

$$G^i - 2\alpha^2\tau b^i = \bar{G}^i_{\bar{\alpha}} + \frac{1}{2\bar{b}^2}\left(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i\right) + \theta y^i,$$
$$s_{ij} = 0$$
$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i)$$
$$b_{i|j} = 2\tau\{(1 - 2b^2)a_{ij} + 3b_ib_j\}$$

Where $b_{i|j}$ denotes the coeffcients of the covariant derivative of β with respect to α.

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